Impedance, Bandwidth, and Q of Antennas[1]

Arthur D. Yaghjian* and Steven R. Best
Air Force Research Laboratory, Hanscom AFB, MA 01731 USA

I. INTRODUCTION

The primary purpose of this paper is twofold: first, to derive an approximate expression for the bandwidth of a tuned antenna in terms of its input impedance that holds at every frequency, that is, throughout the entire antiresonant as well as resonant frequency ranges of the antenna; and second, to relate this expression for bandwidth to the antenna quality factor Q. The definition of stored energy that we use to define the Q of an antenna is similar to that of previous authors [1]–[7]. However, the approximate expression for the bandwidth and its relationship to Q are both more generally applicable and more accurate than previous formulas. The validity and accuracy of the expressions are confirmed by the numerical solutions to straight-wire and wire-loop, lossy and lossless tuned antennas over a wide enough range of frequencies to cover several resonant and antiresonant frequency bands.

It is shown that the matched VSWR bandwidth is the more fundamental measure of bandwidth than conductance bandwidth because it exists in general for all frequencies at which an antenna is tuned. We also find that the Foster reactance theorem does not hold at all frequencies for antennas (whether or not the antenna is lossless) [8, sec. 8-4]. Although the general formula we derive for the bandwidth of an antenna involves the frequency derivative of resistance as well as the frequency derivative of reactance, quite remarkably, the half-power matched VSWR bandwidth of a general tuned lossy or lossless antenna is proven to approximately equal $2/Q$ for all frequencies if $Q \gtrsim 4$.

II. PRELIMINARY DEFINITIONS

Consider a general transmitting antenna composed of linear electromagnetic materials and fed by a "feed line" that carries just one propagating mode at the time-harmonic ($e^{j\omega t}$) frequency $\omega > 0$. The propagating mode in the feed line can be characterized at a reference plane S_0 (which separates the antenna from its shielded power supply) by a complex voltage and current. Let the antenna be tuned at a frequency ω_0 with a series reactance $X_s(\omega)$ comprised of either a series inductance L_s or series capacitance C_s, where L_s and C_s are independent of frequency, to make the total reactance $X_0(\omega) = X(\omega) + X_s(\omega)$ equal to zero at $\omega = \omega_0$. Equations for the tuned antenna relating the impedance $Z_0(\omega)$ to resistance $R_0(\omega)$, reactance $X_0(\omega)$, voltage $V_0(\omega)$ and current $I_0(\omega)$, and the definitions of voltage and current in terms of the incident $a_0(\omega)$ and emergent $b_0(\omega)$ coefficients of the propagating mode in the feed line are given by

$$Z_0 = R_0 + jX_0 = V_0/I_0, \quad V_0 = a_0 + b_0, \quad I_0 = (a_0 - b_0)/Z_c \tag{1}$$

where Z_c is the characteristic impedance of the feed line, which can be chosen independent of frequency. The tuned frequency ω_0, at which $X_0(\omega_0) = 0$, defines a *resonant frequency* of the antenna if $X_0'(\omega_0) > 0$ and an *antiresonant frequency* of the antenna if $X_0'(\omega_0) < 0$.

III. FORMULAS FOR THE BANDWIDTH OF ANTENNAS

The bandwidth of an antenna tuned to zero reactance is usually defined in one of two ways. The first way defines what is commonly called the *conductance bandwidth* and the second way defines what is commonly called the matched *VSWR bandwidth.* The conductance bandwidth for an antenna tuned at a frequency ω_0 is defined as the difference between the two frequencies at which the power accepted by the antenna, excited by a constant value of voltage V_0, is a given fraction of the power accepted at the frequency ω_0. With the help of (1), the conductance at a frequency ω of an antenna tuned at the frequency ω_0 can be written

$$G_0(\omega) = \mathrm{Re}[1/Z_0(\omega)] = R_0(\omega)/[R_0^2(\omega) + X_0^2(\omega)]. \tag{2}$$

We can immediately see from (2) that there is a problem with using conductance bandwidth, namely, that the derivative of $G_0(\omega)$ evaluated at ω_0 equals $G_0'(\omega_0) = -R_0'(\omega_0)/R_0^2(\omega_0)$ and thus it is not zero at ω_0 unless $R_0'(\omega_0) = 0$. This means that in general the conductance will not reach a maximum at the frequency ω_0. Moreover, in antiresonant frequency ranges where both the resistance and reactance of the antenna are changing rapidly with frequency, the conductance may not possess a well-defined maximum and consequently the conductance bandwidth may not exist in these antiresonant frequency ranges. Fortunately, the matched VSWR bandwidth does

[1]This work was supported by the U.S. Air Force Office of Scientific Research (AFOSR).

0-7803-7846-6/03/$17.00 ©2003 IEEE

not suffer from these limitations.

The matched VSWR bandwidth is the difference in frequencies on either side of ω_0 at which the VSWR equals a constant σ or, equivalently, at which the magnitude squared of the reflection coefficient $|\Gamma_0(\omega)|^2$ equals $\alpha = (\sigma-1)^2/(\sigma+1)^2$ (the constant α is assumed chosen $\leq 1/2$), provided the characteristic impedance Z_c of the feed line equals $Z_0(\omega_0) = R_0(\omega_0)$. Then $|\Gamma_0(\omega)|^2$ can be expressed from (1) as

$$|\Gamma_0(\omega)|^2 = \left|\frac{b_0(\omega)}{a_0(\omega)}\right|^2 = \frac{X_0^2(\omega) + [R_0(\omega) - R_0(\omega_0)]^2}{X_0^2(\omega) + [R_0(\omega) + R_0(\omega_0)]^2}. \quad (3)$$

Both $|\Gamma_0(\omega)|^2$ and its derivative with respect to ω are zero at ω_0. Consequently, $|\Gamma_0(\omega)|^2$ has a minimum at ω_0 for all values of the frequency ω_0 at which the antenna is tuned $[X_0(\omega_0) = 0]$ and matched to the feed line $[Z_c = R_0(\omega_0)]$. This means that the matched VSWR bandwidth determined by $|\Gamma_0(\omega)|^2 = \alpha$ or from (3)

$$X_0^2(\omega) + [R_0(\omega) - R_0(\omega_0)]^2 = 4\beta R_0(\omega_0) R_0(\omega), \qquad \beta = \alpha/(1-\alpha) \leq 1 \quad (4)$$

unlike the conductance bandwidth, exists in general at all frequencies, that is, throughout both the antiresonant $[X_0'(\omega_0) < 0]$ and resonant $[X_0'(\omega_0) > 0]$ frequency ranges. Therefore, the matched VSWR bandwidth is a more fundamental, universally applicable definition of bandwidth for a general antenna than conductance bandwidth.

To find the two frequencies $\omega = \omega_0 + \Delta\omega_\pm$ satisfying (4), expand $R_0(\omega_0 + \Delta\omega_\pm)$ and $X_0(\omega_0 + \Delta\omega_\pm)$ in a Taylor series about ω_0 and insert these expansions into (4) to get

$$|Z_0'(\omega_0)|^2 (\Delta\omega_\pm)^2 \approx 4\beta R_0(\omega_0) \left[R_0(\omega_0) + R_0'(\omega_0)\Delta\omega_\pm\right] \quad (5)$$

under the assumptions that $\beta|R_0(\omega_0) R_0''(\omega_0)|/|Z_0'(\omega_0)|^2 \ll 1$ and the $O\left[(\Delta\omega_\pm)^3\right]$ terms are negligible. These latter two assumptions are generally satisfied if $|\Delta\omega_\pm/\omega_0| \ll 1$. The solution to this quadratic equation (5) for $\Delta\omega_\pm$ is given by

$$\Delta\omega_\pm \approx \frac{2R_0(\omega_0)}{|Z_0'(\omega_0)|}\left[\beta R_0'(\omega_0)/|Z_0'(\omega_0)| \pm \sqrt{\beta + |\beta R_0'(\omega_0)/Z_0'(\omega_0)|^2}\right]. \quad (6)$$

The ratio $\beta|R_0'(\omega_0)/Z_0'(\omega_0)|$ is less than or equal to unity and, in addition, for most tuned antennas, $|R_0'(\omega_0)| \lesssim |X_0'(\omega_0)|$ throughout the resonant and antiresonant frequency ranges. Therefore, the square root in (6) can be approximated by $\sqrt{\beta}$, so that the fractional matched VSWR bandwidth $FBW_V(\omega_0)$ is given by

$$\boxed{FBW_V(\omega_0) = \frac{\Delta\omega_+ - \Delta\omega_-}{\omega_0} \approx \frac{4\sqrt{\beta}\, R_0(\omega_0)}{\omega_0 |Z_0'(\omega_0)|}, \qquad \sqrt{\beta} = \frac{\sigma - 1}{2\sqrt{\sigma}} \quad (7)}$$

which holds for tuned antennas under the condition that $|\Delta\omega_\pm/\omega_0| \ll 1$ or $FBW_V(\omega_0) \ll 1$. For most tuned antennas, this condition can be relaxed to $FBW_V(\omega_0) \lesssim \sqrt{\beta}/2$. For half-power VSWR bandwidth, $\alpha = 1/2$ ($\sigma = 5.828$) and $\sqrt{\beta} = 1$. As far as we know, (7) is a general result for antennas that has not been derived or published previously.

IV. IMPEDANCE AND BANDWIDTH IN TERMS OF FIELDS AND Q

The formula for matched VSWR bandwidth given in (7) requires the derivative of impedance with respect to frequency evaluated at $\omega = \omega_0$, that is, $Z_0'(\omega_0) = R_0'(\omega_0) + jX_0'(\omega_0)$. However, as we shall see, an explicit expression for $R_0'(\omega_0)$ in terms of fields is not needed in the derivation of Q and its relationship to the bandwidth of the tuned antenna. On the other hand, the evaluation of the frequency derivative of the reactance, $X_0'(\omega_0)$, in terms of the electromagnetic fields of the antenna is crucial to the derivation of Q and its relationship to bandwidth.

A. Expression for the Frequency Derivative of Reactance

A convenient expression for $X_0'(\omega_0)$ can be derived by combining Maxwell's equations with the frequency derivative of Maxwell's equations to get

$$|I_0|^2 X_0'(\omega_0) = \lim_{r\to\infty}\left[\mathrm{Re}\int\limits_{\mathcal{V}_o(r)} (\mathbf{B}\cdot\mathbf{H}^* + \boldsymbol{\mathcal{D}}^*\cdot\mathbf{E})\, d\mathcal{V} - 2\epsilon_0 r \int\limits_{4\pi} |\mathbf{F}|^2 d\Omega\right]$$
$$+ \omega_0 \mathrm{Re}\int\limits_{\mathcal{V}_a} \left(\mathbf{B}_{I_0}'\cdot\mathbf{H}^* - \mathbf{B}\cdot\mathbf{H}_{I_0}'^* + \mathbf{E}\cdot\boldsymbol{\mathcal{D}}_{I_0}'^* - \boldsymbol{\mathcal{D}}^*\cdot\mathbf{E}_{I_0}'\right) d\mathcal{V} + \frac{2}{Z_f}\mathrm{Im}\int\limits_{4\pi} \mathbf{F}_{I_0}'\cdot\mathbf{F}^* d\Omega. \quad (8)$$

The usual electric and magnetic vectors are denoted by $(\mathbf{E},\mathbf{D})$ and $(\mathbf{B},\mathbf{H})$, respectively, with

$\mathcal{D} = \mathbf{D} + \mathbf{J}/(j\omega)$, the vector $\mathbf{J}$ being the current density. The far electric field $\mathbf{F}(\theta, \phi) = \lim_{r\to\infty} re^{jkr}\mathbf{E}(\mathbf{r})$. The primes indicate derivatives with respect to ω evaluated at the tuned frequency ω_0, and the subscript "I_0" indicates that the feed-line current I_0 is held constant with frequency during the indicated differentiation. The volume $\mathcal{V}_o(r)$ is $\mathcal{V}_o$ capped by a sphere of radius r surrounding the antenna system. Each of the two integral terms inside the square brackets of (8) approaches an infinite value as $r \to \infty$, but together they approach a finite value because all the other terms in (8) are finite. As $r \to \infty$, the second integral term inside the square brackets, the one with $|\mathbf{F}|^2$, subtracts the infinite energy in the radiation fields from the infinite total energy in the fields to leave a finite energy involving "static" and "induction" fields.

From Poynting's theorem and the fact that $X_0(\omega_0) = 0$, we also have that $|I_0|^2 X_0(\omega_0)/\omega_0 = \lim_{r\to\infty} \mathrm{Re} \int_{\mathcal{V}_o(r)} (\mathbf{B} \cdot \mathbf{H}^* - \mathcal{D}^* \cdot \mathbf{E})\, d\mathcal{V} = 0$. Therefore, (8) can be rewritten as

$$|I_0|^2 X_0'(\omega_0) = 4[W(\omega_0) + W_{\mathcal{L}}(\omega_0) + W_{\mathcal{R}}(\omega_0)] \tag{9}$$

with

$$W(\omega_0) = W_m(\omega_0) + W_e(\omega_0) = 2W_m(\omega_0) = 2W_e(\omega_0) \tag{10}$$

$$\left\{ \begin{array}{c} W_m(\omega_0) \\ W_e(\omega_0) \end{array} \right\} = \frac{1}{4} \lim_{r\to\infty} \left[\int\limits_{\mathcal{V}_o(r)} \left\{ \begin{array}{c} \mu_r |\mathbf{H}|^2 \\ \epsilon_r |\mathbf{E}|^2 \end{array} \right\} d\mathcal{V} - \epsilon_0 r \int\limits_{4\pi} |\mathbf{F}|^2 d\Omega \right] \tag{11}$$

$$W_{\mathcal{L}}(\omega_0) = \frac{\omega_0}{2} \mathrm{Im} \int\limits_{\mathcal{V}_a} \left(\mu_i \mathbf{H}_{I_0}' \cdot \mathbf{H}^* + \epsilon_i \mathbf{E}_{I_0}' \cdot \mathbf{E}^*\right) d\mathcal{V} + \frac{\omega_0}{4} \int\limits_{\mathcal{V}_a} \left(\mu_r' |\mathbf{H}|^2 + \epsilon_r' |\mathbf{E}|^2\right) d\mathcal{V} \tag{12}$$

$$W_{\mathcal{R}}(\omega_0) = \frac{1}{2Z_f} \mathrm{Im} \int\limits_{4\pi} \mathbf{F}_{I_0}' \cdot \mathbf{F}^* d\Omega \tag{13}$$

in which we have assumed the constitutive relations $\mathbf{B} = (\mu_r - j\mu_i)\mathbf{H}$ and $\mathcal{D} = (\epsilon_r - j\epsilon_i)\mathbf{E}$. Note that magnetic and electric energies $W_m(\omega)$ and $W_e(\omega)$ at any frequency ω can be defined by (11) evaluated at any frequency ω instead of ω_0. Each of these energies is finite, and the reactance at any frequency ω can thus be written in terms of these finite energies as $X_0(\omega) = 4\omega[W_m(\omega) - W_e(\omega)]/|I_0|^2$, which equals zero at $\omega = \omega_0$.

Although $W_m(\omega)$ and $W_e(\omega)$ are not the energies in quasi-static fields and the antenna material may be highly dispersive, it still seems appropriate to refer to them as "stored" magnetic and electric energies because, as shown below, they define a Q that is inversely proportional to the bandwidth of a tuned antenna if $Q \gg 1$ ($Q \gtrsim 4$).

The energies in (9) denoted by $W_{\mathcal{L}}$ and $W_{\mathcal{R}}$ are associated with the power dissipated by the antenna in the form of power loss and power radiated, respectively. Unlike the power loss and radiated, however, their sum can be positive, negative, or zero and $X_0'(\omega_0)$ in (9) can be positive, negative, or zero. *Therefore, the Foster reactance theorem, which says that $X_0'(\omega)$ for a one-port lossless network is always positive, does not hold at all frequencies for antennas* [8, sec. 8-4].

B. Definition and Exact Expressions of Q

The quality factor $Q(\omega_0)$ for an antenna tuned to have zero reactance at the frequency ω_0 [$X_0(\omega_0) = 0$] can be defined [1]–[7] analogously to the quality factor for resonant circuits in terms of the "stored" energy $W(\omega_0)$ and the accepted power $P_A(\omega_0)$, which equals the total power dissipated as power radiated by the antenna plus power loss in the antenna material:

$$Q(\omega_0) = \omega_0 W(\omega_0)/P_A(\omega_0). \tag{14}$$

Formulas for the "stored" energy $W(\omega_0)$ in terms of fields are given by means of (10)–(11) and the power accepted by the antenna can be expressed as $P_A = |I_0|^2 R_0/2$. Thus (14) allows (9) to be rewritten in the form

$$X_0'(\omega_0) = 2R_0(\omega_0)Q(\omega_0)/\omega_0 + 4[W_{\mathcal{L}}(\omega_0) + W_{\mathcal{R}}(\omega_0)]/|I_0|^2 \tag{15}$$

so that $Q(\omega_0)$ can be expressed as

$$\boxed{Q(\omega_0) = \frac{\omega_0}{2R_0(\omega_0)} X_0'(\omega_0) - \frac{2\omega_0}{|I_0|^2 R_0(\omega_0)} [W_{\mathcal{L}}(\omega_0) + W_{\mathcal{R}}(\omega_0)]. \qquad (16)}$$

The expressions on the right-hand sides of (14) and (16) are very different in form, yet they are exact and thus produce the same value of $Q(\omega_0)$. Especially note that the $Q(\omega_0)$ in (16)

differs from the conventional formula for the quality factor $Q_c(\omega_0) = \omega_0 X_0'(\omega_0)/[2R_0(\omega_0)]$ by the amount $2\omega_0 R_0(\omega_0)[W_\mathcal{L}(\omega_0) + W_\mathcal{R}(\omega_0)]/|I_0|^2$. This Q_c is commonly used to determine the quality factor and the bandwidth ($1/Q_c$ for half-power conductance bandwidth and $2/Q_c$ for half-powerVSWR bandwidth) of tuned antennas.

C. Approximate Expression for Q and Its Relationship to Bandwidth

We can estimate the dissipation energy, $W_\mathcal{L}(\omega_0) + W_\mathcal{R}(\omega_0)$, in (16) to get an approximate expression for $Q(\omega_0)$ that can be immediately related to the bandwidth of the tuned antenna. As mentioned above, the sum $W_\mathcal{L}(\omega_0) + W_\mathcal{R}(\omega_0)$ is energy associated with the power loss and power radiated by the tuned antenna. Away from antiresonant frequency ranges of most tuned antennas, $X_0'(\omega_0) > 0$ and $|R_0'(\omega_0)| \ll X_0'(\omega_0)$, and if the Q is large, the power loss and power radiated can both be approximated by ohmic loss in a resistor of a series RLC circuit. Evaluating $W_\mathcal{L}(\omega_0) + W_\mathcal{R}(\omega_0)$ for such a series RLC circuit reveals that it's value is small enough to make the second term on the right-hand side of (16) negligible compared to the first. Therefore, away from antiresonant frequency ranges and for large Q (say $Q \gtrsim 4$), $Q(\omega_0) \approx \omega_0 X_0'(\omega_0)/[2R_0(\omega_0)] \approx \omega_0|Z_0'(\omega_0)|/[2R_0(\omega_0)]$.

At an antiresonant frequency ω_0, most tuned antennas with a large Q can be approximated by a parallel RLC circuit. An evaluation of $W_\mathcal{L}(\omega_0) + W_\mathcal{R}(\omega_0)$ for such a tuned parallel RLC circuit reveals that $X_0'(\omega_0) - 4[W_\mathcal{L}(\omega_0) + W_\mathcal{R}(\omega_0)]/|I_0|^2 \approx |Z_0'(\omega_0)|$. Therefore,

$$Q(\omega_0) \approx \frac{\omega_0}{2R_0(\omega_0)}|Z_0'(\omega_0)| \tag{17}$$

for all ω_0. These approximate values for $Q(\omega_0)$ in (17) are positive for all ω_0 and, in addition, (17) holds in general only for large Q ($Q \gtrsim 4$). Therefore, (17) does not apply if the Q is too small and especially if the antenna is so dominated by negative μ_r and ϵ_r materials that its $Q(\omega_0)$ is negative. Such unusual antennas would be highly dispersive and could not be well approximated by conventional series or parallel RLC circuits. Ordinarily, μ_r and ϵ_r are positive and it can be proven that the Q of an antenna increases extremely rapidly as the maximum dimension of the source region of the antenna is decreased while maintaining its frequency, efficiency, and far-field pattern (thus confirming that supergain above a few dB is impractical).

Comparing the approximate formula for the quality factor $Q(\omega_0)$ in (17) with the approximate formula for the matched VSWR fractional bandwidth $FBW_V(\omega_0)$ in (7), one finds

$$\boxed{Q(\omega_0) \approx \frac{2\sqrt{\beta}}{FBW_V(\omega_0)} \approx \frac{\omega_0}{2R_0(\omega_0)}|Z_0'(\omega_0)| \qquad (18)}$$

provided $Q(\omega_0) \gtrsim 4$. An approximate expression for Q similar to (18) was derived by Geyi *et al.* [5, eq. (65)], but with $|Z_0'(\omega_0)|$ replaced by $|X_0'(\omega_0)|$ (and $\beta = 1$). Such an expression would not produce an accurate approximation to Q and bandwidth in antiresonant frequency ranges.

V. NUMERICAL RESULTS

Exact VSWR bandwidths are computed from the magnitude of the reflection coefficient versus frequency curves obtained from the numerical solutions to tuned, thin straight-wire and wire-loop antennas ranging in length from a small fraction of a wavelength to many wavelengths. The exact values of Q for these antennas are computed from the general expression (16) derived for the Q of tuned antennas. The exact values of VSWR bandwidth and Q are compared to the approximate values obtained from the derived approximate formula (18) for VSWR bandwidth and Q. These numerical comparisons confirm that the approximate formula (18) for VSWR bandwidth and Q of a tuned antenna gives much more accurate values in antiresonant frequency ranges than the conventional quality factor Q_c [given below (16)].

REFERENCES

[1] L.J. Chu, "Physical limitations of omni-directional antennas," JAP, **19**, 1163–1175, 1948.

[2] R.E. Collin and S. Rothschild, "Evaluation of antenna Q," *IEEE T-AP*, **AP-12**, 23–27, 1964.

[3] R.L. Fante, "Quality factor of general ideal antennas," *IEEE T-AP*, **AP-17**, 151–155, 1969.

[4] J.S. McLean, "A re-examination .. Q of electrically small antennas," *IEEE T-AP*, **44**, 672–676, 1996.

[5] W. Geyi, P. Jarmuszewski, and Y. Qi, "The Foster reactance theorem for antennas and radiation Q," *IEEE T-AP*, **48**, 401–408, 2000.

[6] J.C.-E. Sten *et al.*, "Quality factor of an electrically small antenna radiating close to a conducting plane," *IEEE T-AP*, **49**, 829–837, 2001.

[7] R.F. Harrington, "Effect of antenna size on gain, bandwidth, and efficiency," *J. Res. NBS*, **64D**, 1–12, 1960.

[8] R.F. Harrington, *Time-Harmonic Electromagnetic Fields*, New York: McGraw-Hill, 1961.